# Asymptotics of the eigenvalues of elliptic systems with fast oscillating coefficients

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**Abstract**

We consider singularly perturbed second order elliptic system in the whole space with fast oscillating coefficients. We construct the complete asymptotic expansions for the eigenvalues converging to the isolated ones of the homogenized system, as well as the complete asymptotic expansions for the associated eigenfunctions.

# Introduction

Many works are devoted to the studying of the asymptotic behaviour of the elliptic operators in bounded domains (see, for instance, [1, 2] and the references therein). The similar question for unbounded domains are studied much less. Recently quite intensive study of the such problems has been initiated (see [3, 4, 5, 6] and the references therein). One of the interesting question concerns the behaviour of the spectrum of the mentioned operators in unbounded domains treated as the operators in $L_2$. The one-dimensional case was studied in [7, 8, 9, 10]. There we considered the operators whose coefficients depended on slow and fast variable. The dependence of the slow one was supposed to localized on the finite interval, i.e., at infinity the coefficients depended either on fast variable or were constant. We have studied in detail the asymptotic behavior of continuous and point spectrum and constructed the asymptotic expansions for the eigenvalues and the associated eigenfunctions, as well as for the edges of the zones of the continuous spectrum.

In the present work we generalize partially the results of the cited papers to the multi-dimensional case. Namely, we consider a second order elliptic system in a multi-dimensional space with fast oscillating coefficients. The coefficients depend on slow and fast variable and are periodic w.r.t. the fast one, while they are uniformly bounded together with all their derivatives w.r.t. the slow variable. The main result of the paper is the asymptotic expansions for the eigenvalues of the perturbed system converging to the isolated eigenvalues of the homogenized one. Moreover, we construct the complete asymptotic expansions for the corresponding eigenfunctions. We also note that the similar elliptic system has already been studied in [3], and the leading terms of the asymptotic expansion for the resolvent were constructed. It has been also shown in [3] that basic operators of the

The work is supported in parts by RFBR (07-01-00037). The author is also supported by *Marie Curie International Fellowship* within 6th European Community Framework Programm (MIF1-CT-2005-006254). The author gratefully acknowledges the support from Deligne 2004 Balzan prize in mathematics.

mathematical physics are particular cases of this elliptic system; the great number of interesting examples has been adduced in [4, 5]. The results of our work are applicable for all these examples; in the case of the examples in [4, 5] we also allow the coefficients to depend on the slow variable.

# 1 Formulation of the problem and main results

Let $Y$ be a Banach space. By $W_2^k(\mathbb{R}^d;Y)$ and $W_2^k(\mathbb{R}^d;Y)$, $d \geqslant 1$, we denote the Sobolev space of the functions defined on $\mathbb{R}^d$ with values in $Y$ possessing the finite norms

$$\|\mathbf{u}\|_{W_\infty^k(\mathbb{R}^d;Y)} := \max_{\substack{\alpha\in\mathbb{Z}_+^d\\ |\alpha|\leqslant k}} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \left\| \frac{\partial^{|\alpha|}\mathbf{u}}{\partial x^a}\right\|_Y, \quad \|\mathbf{u}\|^2_{W_2^k(\mathbb{R}^d;Y)} = \sum_{\substack{\alpha\in\mathbb{Z}_+^d\\ |\alpha|\leqslant k}} \int\limits_{\mathbb{R}^d} \left\| \frac{\partial^{|\alpha|}\mathbf{u}}{\partial x^a}\right\|_Y^2 dx.$$

If $k = 0$, we will employ the notations $L_\infty(\mathbb{R}^d;Y) := W_\infty^0(\mathbb{R}^d;Y)$, $L_2(\mathbb{R}^d;Y) := W_2^0(\mathbb{R}^d;Y)$. We denote $\mathcal{W}(\mathbb{R}^d;Y) := \bigcap\limits_{i=1}^{\infty} W_\infty^k(\mathbb{R}^d;Y)$.

In the space $\mathbb{R}^d$ we select a periodic lattice with the elementary cell $\square$. We will employ the symbol $C^\gamma_{per}(\overline{\square})$ to indicate the space of $\square$-periodic functions having finite Hölder norm $\|\cdot\|_{C^\gamma_{per}(\overline{\square})} := \|\cdot\|_{C^\gamma(\overline{\square})}$. In $\mathbb{R}^d$ we introduce the Cartesian coordinates $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d)$. We will often treat a $\square$-periodic w.r.t. $\xi$ vector-function $\mathbf{f} = \mathbf{f}(x,\xi)$ as mapping points $x \in \mathbb{R}^d$ into the function $f = f(x,\cdot)$. It will allow us to speak about the belonging of $\mathbf{f}(x,\xi)$ to $W_\infty^k(\mathbb{R}^d; C^\gamma_{per}(\overline{\square}))$ and $W_2^k(\mathbb{R}^d; C^\gamma_{per}(\overline{\square}))$.

Let $A = A(x,\xi) \in \mathcal{W}(\mathbb{R}^d; C^{1+\beta}_{per}(\overline{\square}))$ be a matrix-valued function of the size $m\times m$, $m \geqslant 1$, $\beta \in (0,1)$. We assume that it is hermitian and satisfies the uniform in $(x,\xi) \in \mathbb{R}^{2d}$ estimate

$$c_1 E_m \leqslant A(x,\xi) \leqslant c_2 E_m,$$

where $E_m$ is $m \times m$ unit matrix. By $B = B(\zeta)$ we denote the matrix-valued function $B(\zeta) = \sum_{i=1}^d B_i\zeta_i$, where $\zeta = (\zeta_1, \ldots, \zeta_d)$, and $B_i$ are constant $m \times n$ matrices, and $m \geqslant n$, $\operatorname{rank} B(\zeta) = n$, $\zeta \neq 0$. Let $V = V(x,\xi) \in \mathcal{W}(\mathbb{R}^d; C^\beta_{per}(\overline{\square}))$, $a_i = a_i(x,\xi) \in \mathcal{W}(\mathbb{R}^d; C^{1+\beta}_{per}(\overline{\square}))$, $b_i = b_i(x) \in \mathcal{W}(\mathbb{R}^d)$ are matrix-valued functions of the sizes $n\times n$, and the matrix $V$ is assumed to be hermitian. The entries of all the matrices are supposed to be complex-valued. By $\varepsilon$ we denote a small positive parameter, and given function $f(x,\xi)$ we let $f_\varepsilon(x) := f(x, x/\varepsilon)$.

The perturbed operator is introduced as

$$\mathcal{H}_\varepsilon := B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x,\partial) + V_\varepsilon$$

in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with the domain $W_2^2(\mathbb{R}^d;\mathbb{C}^n)$. Here

$$B(\partial) := \sum_{i=1}^d B_i\partial_i, \quad B(\partial)^* := -\sum_{i=1}^d B_i^*\partial_i,$$

$$a_\varepsilon(x,\partial) := a\left(x, \frac{x}{\varepsilon}, \partial\right), \quad a(x,\xi,\zeta) := \sum_{i=1}^{d} \big(a_i(x,\xi)\zeta_i b_i(x) - b_i^*(x)\zeta_i a_i^*(x,\xi)\big),$$

where $\partial = (\partial_1, \ldots, \partial_d)$, $\partial_i$ is the derivative w.r.t. $x_i$, the superscript $^*$ indicates hermitian conjugation. It was shown in [3] that the operator $\mathcal{H}_\varepsilon$ is self-adjoint and lower-semibounded uniformly in $\varepsilon$, and the homogenized operator was obtained. Let us describe the latter.

Let $\Lambda_0 = \Lambda_0(x,\xi), \Lambda_1 = \Lambda_1(x,\xi)$ be the matrices of the size $n \times n$ and $n \times m$, respectively, being $\square$-periodic w.r.t. $\xi$ solutions of the equations

$$\begin{aligned} &B(\partial_\xi)^* A(x,\xi) B(\partial_\xi)\Lambda_0(x,\xi) - \sum_{i=1}^{d} b_i^*(x) \frac{\partial a_i^*}{\partial \xi_i}(x,\xi) = 0, \quad (x,\xi) \in \mathbb{R}^{2d}, \\ &B(\partial_\xi)^* A(x,\xi)\big(B(\partial_\xi)\Lambda_1(x,\xi) + E_m\big) = 0, \qquad\qquad\qquad (x,\xi) \in \mathbb{R}^{2d}, \end{aligned} \tag{1.1}$$

and satisfying the conditions

$$\int\limits_{\square} \Lambda_i(x,\xi)\,\mathrm{d}\xi = 0, \quad x \in \mathbb{R}^d, \qquad i = 0,1. \tag{1.2}$$

Here $\partial_\xi = \left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_d}\right)$. It was established in [3] that the problems (1.1), (1.2) are uniquely solvable and $\Lambda_i \in W_\infty^1(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$. The homogenized operator $\mathcal{H}_0$ was determined as follows

$$\begin{aligned} &\mathcal{H}_0 := B(\partial)^* A_2 B(\partial) + A_1(x,\partial) + A_0, \\ &A_2(x) := \frac{1}{|\square|} \int\limits_{\square} A(x,\xi)\big(B(\partial_\xi)\Lambda_1(x,\xi) + E_m\big)\,\mathrm{d}\xi, \\ &A_1(x,\partial) := \frac{1}{|\square|} B(\partial)^* \int\limits_{\square} A(x,\xi) B(\partial_\xi)\Lambda_0(x,\xi)\,\mathrm{d}\xi \\ &\qquad + \left( \frac{1}{|\square|} \int\limits_{\square} \big(B(\partial_\xi)\Lambda_0(x,\xi)\big)^* A(x,\xi)\,\mathrm{d}\xi \right) B(\partial) + \frac{1}{|\square|} \int\limits_{\square} a(x,\xi,\partial)\,\mathrm{d}\xi, \\ &A_0(x) := -\frac{1}{|\square|} \int\limits_{\square} \big(B(\partial_\xi)\Lambda_0(x,\xi)\big)^* A(x,\xi) B(\partial_\xi)\Lambda_0(x,\xi)\,\mathrm{d}\xi + \frac{1}{|\square|} \int\limits_{\square} V(x,\xi)\,\mathrm{d}\xi, \end{aligned}$$

and considered as an operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with the domain $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$. It was shown that this operator is self-adjoint and lower-semibounded.

Let $\lambda_0$ be a $N$-multiple isolated eigenvalue of $\mathcal{H}_0$. It follows from [3, Corollary 1.2] that there exist exactly $N$ eigenvalues $\lambda_\varepsilon^{(i)}$, $i = 1, \ldots, N$, of $\mathcal{H}_\varepsilon$ (counting multiplicity) converging to $\lambda_0$ as $\varepsilon \to +0$. The aim of the paper is to construct the complete asymptotic expansions for these eigenvalues and the associated eigenfunctions. Before presenting our main result, we introduce additional notations.

Let $\boldsymbol{\psi}_0^{(i)}$ be the orthonormalized in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ eigenfunctions associated with $\lambda_0$. We introduce the matrix $T$ with the entries

$$
\begin{aligned}
T_{ij} :=& \frac{1}{|\square|}\big(\mathcal{K}_{-1}(\Lambda_1 B(\partial_x)+\Lambda_0)\boldsymbol{\psi}_0^{(i)}, (\Lambda_1 B(\partial_x)+\Lambda_0)\boldsymbol{\psi}_0^{(j)}\big)_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} \\
&+\frac{1}{|\square|}\big(\boldsymbol{\psi}_0^{(i)}, \mathcal{K}_0(\Lambda_1 B(\partial_x)+\Lambda_0)\boldsymbol{\psi}_0^{(j)}\big)_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} \\
&+\frac{1}{|\square|}\big(\mathcal{K}_0(\Lambda_1 B(\partial_x)+\Lambda_0)\boldsymbol{\psi}_0^{(i)}, \boldsymbol{\psi}_0^{(j)}\big)_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)}, \\
\mathcal{K}_{-1} :=& B(\partial_\xi)^* A B(\partial_x) + B(\partial_x)^* A B(\partial_\xi) + a(x,\xi,\partial_\xi), \\
\mathcal{K}_0 :=& B(\partial_x)^* A B(\partial_x) + a(x,\xi,\partial_x) + V,
\end{aligned}
$$

where $\partial_x := \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d}\right)$, $\partial_\xi := \left(\frac{\partial}{\partial \xi_1},\ldots,\frac{\partial}{\partial \xi_d}\right)$, $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial \xi_i}$ are respectively the partial derivatives w.r.t. $x_i$ and $\xi_i$ for the functions $u = u(x,\xi)$. In formulas given the arguments of all the functions except $\boldsymbol{\psi}_0^{(j)}$ are $(x,\xi)$.

The matrix $T$ being hermitian, there exists a unitary matrix $S_0$ such that the matrix $S_0 T S_0^*$ is diagonal. We denote $\boldsymbol{\Psi}_0^{(i)} := \sum_{j=1}^N S_{ij}^{(0)} \boldsymbol{\psi}_0^{(j)}$, where $S_{ij}^{(0)}$ are the elements of $S_0$. The vector-functions $\boldsymbol{\Psi}_0^{(i)}$ are orthonormalized in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. By $\tau_i$, $i=1,\ldots,N$, we denote the eigenvalues of $T$.

**Theorem 1.1.** *Let the eigenvalues of $T$ be different. Then the eigenvalues $\lambda_\varepsilon^{(i)}$ satisfy the asymptotic expansions*

$$
\lambda_\varepsilon^{(i)} = \lambda_0 + \sum_{j=1}^{\infty} \varepsilon^j \lambda_j^{(i)}, \tag{1.3}
$$

$$
\lambda_1^{(i)} = \tau_i, \tag{1.4}
$$

*where the rest of the coefficients is determined by Lemma 3.2. The eigenfunctions associated with $\lambda_\varepsilon^{(i)}$ can be chosen so that in the norm of $W_2^2(\mathbb{R}^d;\mathbb{C}^n)$ they satisfy the asymptotic expansions*

$$
\boldsymbol{\psi}_\varepsilon^{(i)}(x) = \boldsymbol{\Psi}_0^{(i)}(x) + \sum_{j=1}^{N} \varepsilon^j \boldsymbol{\Psi}_j^{(i)}\left(x, \frac{x}{\varepsilon}\right), \tag{1.5}
$$

$$
\boldsymbol{\Psi}_1^{(i)}(x,\xi) = \big(\Lambda_1(x,\xi)B(\partial_x) + \Lambda_0(x,\xi)\big)\boldsymbol{\Psi}_0^{i}(x) + \boldsymbol{\phi}_1^{(i)}(x), \tag{1.6}
$$

*where $\boldsymbol{\phi}_1^{(i)}$ are given by (3.10), (3.7), (3.13). The rest of the coefficients in (1.5) is determined in Lemma 3.2.*

We stress that the assumption $\tau_i \neq \tau_j$, $i \neq j$, is not essential for the constructing of the asymptotic expansions for the eigenvalues and eigenfunctions of $\mathcal{H}_\varepsilon$. We have used it just to simplify certain technical details. If this assumption does not hold, the technique employed in the proof of Theorem 1.1 allows us to construct the asymptotics for $\lambda_\varepsilon^{(i)}$ and $\boldsymbol{\psi}_\varepsilon^{(i)}$. We also note that this assumption is the general case, if $\lambda_0$ is multiple, and is surely to hold true, if $\lambda_0$ is simple.

## 2 Auxiliary statements

In the present section we prove a series of auxiliary statements.

**Lemma 2.1.** *For any* $\mathbf{u}\in W_2^2(\mathbb{R}^d;\mathbb{C}^n)$ *the uniform in* $\varepsilon$ *estimate*

$$\|\mathbf{u}\|_{W_2^2(\mathbb{R}^d;\mathbb{C}^n)}\leqslant C\varepsilon^{-5}\left(\|\mathcal{H}_\varepsilon\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\|\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\right)$$

*holds true.*

*Proof.* It is sufficient to show the estimate for the vector-functions $\mathbf{u}\in C_0^\infty(\mathbb{R}^d)$ since the latter set is dense in $W_2^2(\mathbb{R}^d;\mathbb{C}^n)$. Throughout the proof we indicate by $C$ inessential constants independent of $\varepsilon$.

We denote $\mathbf{f}:=\mathcal{H}_\varepsilon\mathbf{u}$. Since

$$\begin{aligned}(\mathbf{f},\mathbf{u})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}=&\big(A_\varepsilon B(\partial)\mathbf{u},B(\partial)\mathbf{u}\big)_{L_2(\mathbb{R}^d;\mathbb{C}^m)}\\&+2\operatorname{Re}\sum_{i=1}^d(a_{i,\varepsilon}\partial_i b_i\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\big(V_\varepsilon\mathbf{u},\mathbf{u}\big)_{L_2(\mathbb{R}^d;\mathbb{C}^n)},\end{aligned}$$

it follows from an uniform in $\varepsilon$ inequality

$$C_1\|\nabla\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\leqslant\big(A_\varepsilon B(\partial)\mathbf{u},B(\partial)\mathbf{u}\big)_{L_2(\mathbb{R}^d;\mathbb{C}^m)}\leqslant C_2\|\nabla\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)},$$

proven in Lemma 2.1 in [3] that

$$\|\mathbf{u}\|_{W_2^1(\mathbb{R}^d;\mathbb{C}^n)}\leqslant C\left(\|\mathbf{f}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\|\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\right). \tag{2.1}$$

By the definition of $A_\varepsilon$ and the estimate established we obtain

$$\begin{aligned}\Bigg\|\sum_{i,j=1}^d B_i^*A_\varepsilon B_j\partial_{ij}\mathbf{u}\Bigg\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}&\leqslant\|\mathbf{f}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+C\varepsilon^{-1}\|\mathbf{u}\|_{W_2^1(\mathbb{R}^d;\mathbb{C}^n)}\\&\leqslant C\varepsilon^{-1}\left(\|\mathbf{f}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\|\mathbf{u}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\right).\end{aligned} \tag{2.2}$$

Let $\sum\limits_p\chi_p^2(x)=1$ be a partition of unity for $\mathbb{R}^d$ such that each of cut-off functions obeys an inequality $0\leqslant\|\chi_p\|_{C^2(\operatorname{supp}\chi_p)}\leqslant C$, where the constant $C$ is independent of $p$, and the support of each $\chi_p$ can be shifted inside a fixed bounded domain independent of $p$. We also assume that the number of the functions $\chi_p$ not vanishing at a point $x\in\mathbb{R}^d$ is bounded uniformly in $x\in\mathbb{R}^d$. We denote

$$\mathbf{u}_p(x):=\chi_p\left(\frac{x}{\varepsilon^2}\right)\mathbf{u}(x),\quad\mathbf{f}_p:=-\sum_{i,j=1}^d B_i^*A_\varepsilon B_j\partial_{ij}\mathbf{u}_p.$$

We observe that by (2.1), (2.2)

$$\begin{aligned}\sum_p\|\mathbf{f}_p\|^2_{L_2(\Omega_{p,\varepsilon})}&\leqslant C\varepsilon^{-9}\sum_p\left(\|\chi_p\mathbf{f}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\|\mathbf{u}_p\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\right)\\&=C\varepsilon^{-9}\left(\|\mathbf{f}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}+\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\right),\end{aligned} \tag{2.3}$$

where $\Omega_{p,\varepsilon} := \operatorname{supp}\chi_p\left(\frac{x}{\varepsilon^2}\right)$. The definition of $\chi_p$ yields that $\operatorname{supp} u \subseteq \Omega_{p,\varepsilon}$, and the linear size of $\Omega_{p,\varepsilon}$ is of order $\mathcal{O}(\varepsilon^2)$.

Let $x_p^{(0)}$ be a point in this support. By the smoothness, the matrix $A_\varepsilon$ satisfies the identity

$$A_\varepsilon(x) = A_\varepsilon(x_p^{(0)}) + \varepsilon\widetilde{A}(x,p,\varepsilon), \qquad |\widetilde{A}(x,p,\varepsilon)| \leqslant C, \quad x \in \Omega_{p,\varepsilon},$$

where the constant $C$ is independent of $\varepsilon$, $p$ and $x \in \Omega_{p,\varepsilon}$. Thus,

$$-\sum_{i,j=1}^{d} B_i^* A_\varepsilon(x_p^{(0)}) B_j \partial_{ij}\mathbf{u}_p = \mathbf{f}_p + \varepsilon \sum_{i,j=1}^{d} B_i^* \widetilde{A}_\varepsilon B_j \partial_{ij}\mathbf{u}_p, \quad x \in \Omega_{p,\varepsilon}.$$

The left-hand side of this equation contains a differential operator with constant coefficients that allows us to employ the estimate (10.1) from [11, Ch. IV, §10.1, Th. 10.1] and to obtain

$$\sum_{i,j=1}^{d} \|\partial_{ij}\mathbf{u}_p\|_{L_2(\Omega_{p,\varepsilon};\mathbb{C}^n)} \leqslant C\left(\|\mathbf{f}_p\|_{L_2(\Omega_{p,\varepsilon};\mathbb{C}^n)} + \varepsilon\sum_{i,j=1}^{d} \|\partial_{ij}\mathbf{u}_p\|_{L_2(\Omega_{p,\varepsilon};\mathbb{C}^n)}\right),$$

where the constant $C$ is independent of $\varepsilon$ and $p$. In view of (2.3) we conclude now that

$$\begin{aligned}
&\sum_{i,j=1}^{d} \|\partial_{ij}\mathbf{u}_p\|_{L_2(\Omega_{p,\varepsilon};\mathbb{C}^n)} \leqslant C\|\mathbf{f}_p\|_{L_2(\Omega_{p,\varepsilon};\mathbb{C}^n)},\\
&\sum_{i,j=1}^{d} \|\partial_{ij}\mathbf{u}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)} = \sum_{i,j=1}^{d}\sum_{p} \left\|\chi_p\partial_{ij}\mathbf{u}\right\|^2_{L_2(\Omega_p;\mathbb{C}^n)} \leqslant C\sum_{p}\sum_{i,j=1}^{d} \|\partial_{ij}\mathbf{u}_p\|^2_{L_2(\Omega_p;\mathbb{C}^n)}\\
&\qquad + C\varepsilon^{-8}\sum_{p}\|\mathbf{u}\|^2_{W_2^1(\Omega_p;\mathbb{C}^n)} \leqslant C\varepsilon^{-9}\left(\|\mathbf{f}\|^2_{L_2(\mathbb{R}^d;\mathbb{C}^n)} + \|\mathbf{u}\|^2_{W_2^1(\mathbb{R}^d;\mathbb{C}^n)}\right).
\end{aligned}$$

This inequality lead us to the statement of the lemma. $\square$

**Lemma 2.2.** *Let $\mathbf{f}(x,\cdot) \in C^\beta_{per}(\overline{\square})$ for all $x \in \mathbb{R}^d$. The system*

$$B(\partial_\xi)^* A(x,\xi) B(\partial_\xi)\mathbf{v}(x,\xi) = \mathbf{f}(x,\xi), \quad \xi \in \mathbb{R}^d,$$

*has the $\square$-periodic in $\xi$ solution $\mathbf{v}(x,\cdot) \in C^{2+\beta}_{per}(\overline{\square})$ unique up to a constant (w.r.t. $\xi$) vector, if and only if*

$$\int_{\square} \mathbf{f}(x,\xi)\,\mathrm{d}\xi = 0, \quad x \in \mathbb{R}^d. \tag{2.4}$$

*If this condition is satisfied, there exists the unique solution $\mathbf{v}$ satisfying (2.4), too. If $\mathbf{f} \in W_p^k\big(\mathbb{R}^d; C^\beta_{per}(\overline{\square})\big)$, $p = 2$, $p = \infty$, then $\mathbf{v} \in W_p^k\big(\mathbb{R}^d; C^{2+\beta}_{per}(\overline{\square})\big)$, and the estimate*

$$\|\mathbf{v}\|_{W_p^k(\mathbb{R}^d;C^{2+\beta}_{per}(\overline{\square}))} \leqslant C\|\mathbf{f}\|_{W_p^k(\mathbb{R}^d;C^\beta_{per}(\overline{\square}))}$$

*is valid.*

This lemma is proved completely by analogy with Lemma 2.2 in [3].

**Lemma 2.3.** *For $\lambda$ close to $\lambda_0$, sufficiently small $\varepsilon$, and any $\mathbf{f} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ the representation*

$$(\mathcal{H}_\varepsilon - \lambda)^{-1}\mathbf{f} = \sum_{i=1}^{N} \frac{\mathbf{\Psi}_\varepsilon^{(i)}}{\lambda_\varepsilon^{(i)} - \lambda}(\mathbf{f}, \mathbf{\Psi}_\varepsilon^{(i)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)} + \widetilde{\mathbf{u}}_\varepsilon, \tag{2.5}$$

*holds true, where $\mathbf{\Psi}_\varepsilon^{(i)}$ are the eigenfunctions associated with $\lambda_\varepsilon^{(i)}$ and orthonormalized in $L_2(\mathbb{R}^d; \mathbb{C}^n)$, while the vector-function $\widetilde{\mathbf{u}}_\varepsilon$ satisfies the uniform in $\varepsilon$ and $\lambda$ estimates*

$$\|\widetilde{\mathbf{u}}_\varepsilon\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)} \leqslant C\|\mathbf{f}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}, \quad \|\widetilde{\mathbf{u}}_\varepsilon\|_{W_2^2(\mathbb{R}^d;\mathbb{C}^n)} \leqslant C\varepsilon^{-5}\|\mathbf{f}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{2.6}$$

*Proof.* The representation (2.5) follows from [12, Ch. V, §3.5, Formula (3.21)], where the vector-function $\widetilde{\mathbf{u}}_\varepsilon$ is holomorphic w.r.t. $\lambda$ close to $\lambda_0$ in the norm of $L_2(\mathbb{R}^d; \mathbb{C}^n)$. One can make sure that

$$\widetilde{\mathbf{u}}_\varepsilon = (\mathcal{H}_\varepsilon - \lambda)^{-1}\widetilde{\mathbf{f}}, \quad \widetilde{\mathbf{f}} := \mathbf{f} - \sum_{i=1}^{N}(\mathbf{f}, \mathbf{\Psi}_\varepsilon^{(i)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}\mathbf{\Psi}_\varepsilon^{(i)}. \tag{2.7}$$

Let $\delta$ be a sufficiently small fixed number such that $\sigma_{\mathrm{disc}}(\mathcal{H}_0) \cap \{\lambda : |\lambda - \lambda_0| \leqslant \delta\} = \{\lambda_0\}$. By the convergences $\lambda_\varepsilon^{(i)} \to \lambda_0$ for all $\varepsilon$ small enough we hence have $\operatorname{dist}\big(\sigma_{\mathrm{disc}}(\mathcal{H}_\varepsilon), \{\lambda : |\lambda - \lambda_0| = \delta\}\big) \geqslant \delta/2$. This inequality by [12, Ch. V, §3.5, Formula (3.16)] and (2.7) implies the former of the estimates in (2.6) for $|\lambda - \lambda_0| = \delta$, where the constant $C$ is independent of $\varepsilon$ and $\lambda$. By the maximum modulus principle for the holomorphic functions we conclude that this estimate is valid for $|\lambda - \lambda_0| < \delta$ as well. The former estimate in (2.6) follows now from Lemma 2.1. ☐

# 3 Proof of Theorem 1.1

First we construct formally the asymptotic expansions for the eigenvalues and the eigenfunctions. Then the justification of these asymptotics will be adduced.

We employ the two-scale method [1] in formal constructing. The asymptotics for the eigenvalues of $\mathcal{H}_\varepsilon$ converging to $\lambda_0$ are constructed as the series (1.3), while the asymptotics for the associated eigenfunctions are sought as the series (1.5). The aim of the formal construction is to determine the coefficients of the series (1.3), (1.5). The vector-functions $\Psi_j^{(i)} = \Psi_j^{(i)}(x, \xi)$, $j \geqslant 1$, are sought to be $\square$-periodic in $\xi$, and fast decaying as $|x| \to +\infty$. We will specify their smoothness and behavior at infinity in more details during the constructing.

We substitute the series (1.3), (1.5) into the equation $\mathcal{H}_\varepsilon \boldsymbol{\psi}_\varepsilon^{(i)} = \lambda_\varepsilon^{(i)} \boldsymbol{\psi}_\varepsilon^{(i)}$ and collect the coefficients of the same powers of $\varepsilon$. As a result, we arrive at the series

of the equations

$$B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Psi}_{j+2}^{(i)} = -\mathcal{K}_{-1}\boldsymbol{\Psi}_{j+1}^{(i)} - \mathcal{K}_0\boldsymbol{\Psi}_j^{(i)} + \lambda_0\boldsymbol{\Psi}_j^{(i)} + \sum_{k=1}^{j}\lambda_k^{(i)}\boldsymbol{\Psi}_{j-k}^{(i)}, \quad (x,\xi)\in\mathbb{R}^{2d}, \tag{3.1}$$

where $j \geqslant -1$, $A = A(x,\xi)$, $V = V(x,\xi)$, $\boldsymbol{\Psi}_j^{(i)} = \boldsymbol{\Psi}_j^{(i)}(x,\xi)$, $q \geqslant 1$, $\boldsymbol{\Psi}_j^{(i)} \equiv 0$, $j < 0$. For $j = -1$ the equation (3.1) casts into

$$B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Psi}_1^{(i)} = -\mathcal{K}_{-1}\boldsymbol{\Psi}_0^{(i)}, \quad (x,\xi)\in\mathbb{R}^{2d}.$$

As it follows from (1.1) and the definition of $\mathcal{K}_{-1}$, a $\square$-periodic w.r.t. $\xi$ solution of this equation is given by

$$\boldsymbol{\Psi}_1^{(i)}(x,\xi) = \boldsymbol{\Upsilon}_1^{(i)}(x,\xi) + \boldsymbol{\phi}_1^{(i)}(x), \quad \boldsymbol{\Upsilon}_1^{(i)} := (\Lambda_1 B(\partial) + \Lambda_0)\boldsymbol{\Psi}_i^{(0)}, \tag{3.2}$$

where $\boldsymbol{\phi}_1^{(i)}$ is a vector-function to be determined.

It follows from Lemma 2.2 that $\Lambda_i \in \mathcal{W}(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$, and thus the coefficients of $\mathcal{H}_0$ belong to $\mathcal{W}(\mathbb{R}^d)$. Employing this fact and differentiating the equation $(\mathcal{H}_0 - \lambda_0)\Psi_0^{(i)} = 0$, one can easily make sure that $\boldsymbol{\Psi}_0^{(i)} \in W_2^\infty(\mathbb{R}^d;\mathbb{C}^n)$, where $W_2^\infty(\mathbb{R}^d;\mathbb{C}^n) := \bigcap\limits_{k=1}^{\infty} W_2^k(\mathbb{R}^d;\mathbb{C}^n)$. It implies that $\boldsymbol{\Upsilon}_1^{(i)} \in W_2^\infty(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$.

We substitute (3.2) into (3.1) for $j = 0$ to obtain

$$B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Psi}_2^{(i)} = -\mathcal{K}_{-1}\boldsymbol{\Upsilon}_1^{(i)} - \mathcal{K}_0\boldsymbol{\Psi}_0^{(i)} + \lambda_0\boldsymbol{\Psi}_0^{(i)} - \mathcal{K}_{-1}\boldsymbol{\phi}_1^{(i)}, \quad (x,\xi)\in\mathbb{R}^{2d}. \tag{3.3}$$

In accordance with Lemma 2.2 the equation is uniquely solvable in the class of $\square$-periodic w.r.t. $\xi$ vector-functions, if the solvability condition (2.4) holds true. In view of the identities

$$\begin{aligned}
&\int\limits_{\square} (B(\partial_\xi)\Lambda_0)^* A \,\mathrm{d}\xi = \sum_{i=1}^{d}\int\limits_{\square} a_i b_i \frac{\partial\Lambda_1}{\partial\xi_i}\,\mathrm{d}\xi,\\
&\int\limits_{\square} \big(B(\partial_\xi)\Lambda_0\big)^* A B(\partial_\xi)\Lambda_0 \,\mathrm{d}\xi = -\sum_{i=1}^{d}\int\limits_{\square} a_i b_i \frac{\partial\Lambda_0}{\partial\xi_i}\,\mathrm{d}\xi,
\end{aligned} \tag{3.4}$$

established in [3], it is easy to check that this solvability condition leads us to the equation $(\mathcal{H}_0 - \lambda_0)\boldsymbol{\Psi}_0^{(i)} = 0$, which holds true by the definition of $\Psi_0^{(i)}$. Hence, the vector-function $\boldsymbol{\Psi}_2^{(i)}$ reads as follows

$$\boldsymbol{\Psi}_2^{(i)}(x,\xi) = \boldsymbol{\Upsilon}_2^{(i)}(x,\xi) + (\Lambda_1(x,\xi)B(\partial) + \Lambda_0(x,\xi))\boldsymbol{\phi}_1^{(i)}(x) + \boldsymbol{\phi}_2^{(i)}(x), \tag{3.5}$$

where $\boldsymbol{\phi}_2^{(i)}$ is a vector-function to be determined, and $\boldsymbol{\Upsilon}_2^{(i)}$ are $\square$-periodic w.r.t. $\xi$ solutions to the equation

$$B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Upsilon}_2^{(i)} = -\mathcal{K}_{-1}\boldsymbol{\Upsilon}_1^{(i)} - \mathcal{K}_0\boldsymbol{\Psi}_0^{(i)} + \lambda_0\boldsymbol{\Psi}_0^{(i)}, \quad (x,\xi)\in\mathbb{R}^{2d}, \tag{3.6}$$

and satisfy (2.4). The equation is uniquely solvable, since the right hand side of (3.3) and the vector-function $\mathcal{K}_{-1}\boldsymbol{\phi}_1^{(i)}$ satisfy (2.4). By Lemma 2.2, $\boldsymbol{\Upsilon}_2^{(i)} \in W_2^\infty(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$.

We substitute now (3.2), (3.5) into (3.1) with $j = 1$,

$$\begin{aligned} B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Psi}_3^{(i)} = {} & -\mathcal{K}_{-1}\boldsymbol{\Upsilon}_2^{(i)} - \mathcal{K}_0\boldsymbol{\Upsilon}_1^{(i)} + \lambda_0\boldsymbol{\Upsilon}_1^{(i)} - \mathcal{K}_{-1}(\Lambda_1 B(\partial_x) + \Lambda_0)\boldsymbol{\phi}_1^{(i)} \\ & - \mathcal{K}_0\boldsymbol{\phi}_1^{(i)} + \lambda_0\boldsymbol{\phi}_1^{(i)} + \lambda_1^{(i)}\boldsymbol{\Psi}_0^{(i)} - \mathcal{K}_{-1}\boldsymbol{\phi}_2^{(i)}, \quad (x,\xi) \in \mathbb{R}^{2d}. \end{aligned} \tag{3.7}$$

We write down the solvability condition (2.4) for the equation and take into account (3.4). It leads us to the equation for $\boldsymbol{\phi}_1^{(i)}$,

$$(\mathcal{H}_0 - \lambda_0)\boldsymbol{\phi}_1^{(i)} = \lambda_1^{(i)}\boldsymbol{\Psi}_0^{(i)} - \frac{1}{|\square|}\int\limits_{\square} (\mathcal{K}_{-1}\boldsymbol{\Upsilon}_2^{(i)} + \mathcal{K}_0\boldsymbol{\Upsilon}_1^{(i)})\, \mathrm{d}\xi. \tag{3.8}$$

The right hand side of this equation is an element of $W_2^\infty(\mathbb{R}^d; \mathbb{C}^n)$. Since $\lambda_0$ is an isolated eigenvalue of $\mathcal{H}_0$, the equation is solvable in $W_2^2(\mathbb{R}^d; \mathbb{C}^n)$, if and only if

$$\left(\lambda_1^{(i)}\boldsymbol{\Psi}_0^{(i)} - \frac{1}{|\square|}\int\limits_{\square} (\mathcal{K}_{-1}\boldsymbol{\Upsilon}_2^{(i)} + \mathcal{K}_0\boldsymbol{\Upsilon}_1^{(i)})\, \mathrm{d}\xi, \boldsymbol{\Psi}_0^{(k)}\right)_{L_2(\mathbb{R}^d;\mathbb{C}^n)} = 0, \quad k = 1, \ldots, N. \tag{3.9}$$

If these conditions hold, the solution to (3.8) is defined uniquely up to a linear combination of $\boldsymbol{\Psi}_0^{(i)}$.

**Lemma 3.1.** *The identities*

$$\frac{1}{|\square|}(\mathcal{K}_{-1}\boldsymbol{\Upsilon}_2^{(i)} + \mathcal{K}_0\boldsymbol{\Upsilon}_1^{(i)}, \boldsymbol{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} = \begin{cases} 0, & i \neq k, \\ \tau_i, & i = k, \end{cases}$$

*hold true.*

*Proof.* Integrating by parts and taking into account the equations (1.1), (3.6) and the conditions (1.2), we obtain

$$\begin{aligned} &(\mathcal{K}_{-1}\boldsymbol{\Upsilon}_2^{(i)}, \boldsymbol{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} = (\boldsymbol{\Upsilon}_2^{(i)}, \mathcal{K}_{-1}\boldsymbol{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} \\ &= -(\boldsymbol{\Upsilon}_2^{(i)}, B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Upsilon}_1^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} = -(B(\partial_\xi)^* A B(\partial_\xi)\boldsymbol{\Upsilon}_2^{(i)}, \boldsymbol{\Upsilon}_1^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} \\ &= (\mathcal{K}_{-1}\boldsymbol{\Upsilon}_1^{(i)}, \boldsymbol{\Upsilon}_1^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)} + (\boldsymbol{\Psi}_0^{(i)}, \mathcal{K}_0\boldsymbol{\Upsilon}_1^{(k)})_{L_2(\mathbb{R}^d\times\square;\mathbb{C}^n)}. \end{aligned}$$

By the definition of $T$ and $\boldsymbol{\Psi}_0^{(i)}$ it proves the lemma. $\square$

In view of the definition of $\boldsymbol{\Psi}_i^{(0)}$ the identities (3.9) hold true, if the numbers $\lambda_1^{(i)}$ are chosen in accordance with (1.4). The corresponding solution of the equation (3.8) reads as follows

$$\boldsymbol{\phi}_1^{(i)} = \widetilde{\boldsymbol{\phi}}_1^{(i)} + \sum_{p=1}^{N} S_{ik}^{(1)}\boldsymbol{\Psi}_0^{(k)}, \quad \left(\widetilde{\boldsymbol{\phi}}_1^{(i)}, \boldsymbol{\Psi}_0^{(k)}\right)_{L_2(\mathbb{R}^d;\mathbb{C}^n)} = 0, \quad k = 1, \ldots, N \tag{3.10}$$

where $S_{ip}^{(1)}$ are numbers, and $\widetilde{\boldsymbol{\phi}}_1^{(i)} \in W_2^\infty(\mathbb{R}^d;\mathbb{C}^n)$. The latter belonging can be justified easily by differentiating (3.8). The solution of (3.7) is given by

$$\mathbf{\Psi}_3^{(i)}(x,\xi) = \mathbf{\Phi}_3^{(i)}(x,\xi)+(\Lambda_1(x,\xi)B(\partial_x)+\Lambda_0(x,\xi))\boldsymbol{\phi}_2^{(i)}(x)+\sum_{k=1}^{N} S_{ik}^{(1)}\mathbf{\Upsilon}_2^{(k)}(x,\xi)+\boldsymbol{\phi}_3^{(i)}(x), \tag{3.11}$$

where $\boldsymbol{\phi}_3^{(i)}$ is the vector-function, and $\mathbf{\Phi}_3^{(i)} \in W_2^\infty(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$ is a solution to

$$\begin{aligned} B(\partial_\xi)^* AB(\partial_\xi)\mathbf{\Phi}_3^{(i)} =& -\mathcal{K}_{-1}\mathbf{\Upsilon}_2^{(i)} - \mathcal{K}_0\mathbf{\Upsilon}_1^{(i)} + \lambda_0\mathbf{\Upsilon}_1^{(i)} - \mathcal{K}_{-1}(\Lambda_1 B(\partial_x)+\Lambda_0)\widetilde{\boldsymbol{\phi}}_1^{(i)} \\ & - \mathcal{K}_0\widetilde{\boldsymbol{\phi}}_1^{(i)} + \lambda_0\widetilde{\boldsymbol{\phi}}_1^{(i)} + \lambda_1^{(i)}\mathbf{\Psi}_0^{(i)}, \quad (x,\xi)\in\mathbb{R}^{2d}. \end{aligned}$$

Let us show how to determine $S_{ip}^{(1)}$ and $\lambda_2^{(i)}$. We substitute (3.2), (3.5), (3.10), (3.11) into (3.1) with $j=2$. The solvability condition (2.4) for this equation yields

$$\begin{aligned} (\mathcal{H}_0-\lambda_0)\boldsymbol{\phi}_2^{(i)} =& \lambda_1^{(i)}\sum_{p=1}^{N} S_{ip}^{(1)}\mathbf{\Psi}_0^{(p)} - \frac{1}{|\square|}\sum_{k=1}^{N} S_{ik}^{(1)}\int\limits_{\square}(\mathcal{K}_{-1}\mathbf{\Upsilon}_2^{(k)}+\mathcal{K}_0\mathbf{\Upsilon}_1^{(k)})\,\mathrm{d}\xi \\ & + \lambda_1^{(i)}\widetilde{\boldsymbol{\phi}}_1^{(i)} + \lambda_2^{(i)}\mathbf{\Psi}_0^{(i)} - \mathbf{g}_2^{(i)}, \quad x\in\mathbb{R}^d, \end{aligned} \tag{3.12}$$

$$\mathbf{g}_2^{(i)} := \frac{1}{|\square|}\int\limits_{\square}\big(\mathcal{K}_{-1}\mathbf{\Phi}_3^{(i)} + \mathcal{K}_0\mathbf{\Upsilon}_2^{(i)} + \mathcal{K}_0(\Lambda_1 B(\partial_x)+\Lambda_0)\widetilde{\boldsymbol{\phi}}_1^{(i)}\big)\,\mathrm{d}\xi.$$

The right hand side of the obtained equation belongs to $W_2^\infty(\mathbb{R}^d;\mathbb{C}^n)$. Now we write the solvability condition for (3.12) and take into account Lemma 3.1 and (1.4). It leads us to

$$(\tau_1^{(i)}-\tau_1^{(k)})S_{ik}^{(1)} + \lambda_2^{(i)}\delta_{ik} - (\mathbf{g}_2^{(i)},\mathbf{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)} = 0, \quad k=1,\dots,N,$$

where $\delta_{ik}$ is the Kronecker delta. By the assumption $\tau_i\neq\tau_j$, $i\neq j$, it implies that

$$S_{ik}^{(1)} = \frac{(\mathbf{g}_2^{(i)},\mathbf{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}}{\tau_i-\tau_k}, \quad k\neq i, \qquad \lambda_2^{(i)} = (\mathbf{g}_2^{(i)},\mathbf{\Psi}_0^{(i)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}. \tag{3.13}$$

Without loss of generality we let $S_{ii}^{(1)}=0$. The formulas (1.6) are proven.

The construction of the other terms of the series (1.3), (1.5) is carried out by the same scheme. The results are in the following lemma which can be easily proved by induction.

**Lemma 3.2.** *There exist unique solutions to (3.1) and the uniquely determined numbers $\lambda_j^{(i)}$, which read as follows*

$$\begin{aligned} \mathbf{\Psi}_j^{(i)}(x,\xi) =& \mathbf{\Phi}_j^{(i)}(x,\xi) + (\Lambda_1(x,\xi)B(\partial_x)+\Lambda_0(x,\xi))\boldsymbol{\phi}_{j-1}^{(i)}(x) \\ & + \sum_{k=1}^{N} S_{ik}^{(j-2)}\mathbf{\Upsilon}_k^{(2)}(x,\xi) + \boldsymbol{\phi}_j^{(i)}(x), \end{aligned}$$

$$\boldsymbol{\phi}_j^{(i)}(x) =\widetilde{\boldsymbol{\phi}}_j^{(i)}(x) + \sum_{k=1}^{N} S_{ik}^{(j)} \mathbf{\Psi}_0^{(k)}(x),$$

*where* $\mathbf{\Phi}_j^{(i)} \in W_2^\infty(\mathbb{R}^d; C_{per}^{2+\beta}(\overline{\square}))$ *are the solutions to*

$$\begin{aligned}
B(\partial_\xi)^* A B(\partial_\xi) \mathbf{\Phi}_j^{(i)} =& - \mathcal{K}_{-1} \left( \mathbf{\Phi}_{j-1}^{(i)} + (\Lambda_1 B(\partial_x) + \Lambda_0) \widetilde{\boldsymbol{\phi}}_{j-2}^{(i)} + \sum_{k=1}^{N} S_{ik}^{(j-3)} \mathbf{\Upsilon}_2^{(k)} \right) \\
& - \mathcal{K}_0 \left( \mathbf{\Phi}_{j-2}^{(i)} + (\Lambda_1 B(\partial_x) + \Lambda_0) \widetilde{\boldsymbol{\phi}}_{j-3}^{(i)} + \sum_{k=1}^{N} S_{ik}^{(j-4)} \mathbf{\Upsilon}_2^{(k)} + \widetilde{\boldsymbol{\phi}}_{j-2}^{(i)} \right) \\
& + \sum_{k=0}^{j-2} \lambda_k^{(i)} \mathbf{\Psi}_{j-k-2}, \quad (x,\xi) \in \mathbb{R}^{2d},
\end{aligned}$$

*and satisfy (2.4),* $\lambda_0^{(i)} := \lambda_0$*. The vector-functions* $\widetilde{\boldsymbol{\phi}}_j^{(i)} \in W_2^\infty(\mathbb{R}^d; \mathbb{C}^n)$ *are the solutions to the equations*

$$\begin{aligned}
(\mathcal{H}_0 - \lambda_0) \widetilde{\boldsymbol{\phi}}_j^{(i)} =& \lambda_1^{(i)} \sum_{k=1}^{N} S_{ik}^{(j)} \mathbf{\Psi}_0^{(k)} - \frac{1}{|\square|} \sum_{k=1}^{N} S_{ik}^{(j)} \int\limits_{\square} (\mathcal{K}_{-1} \mathbf{\Upsilon}_2^{(i)} + \mathcal{K}_0 \mathbf{\Upsilon}_1^{(i)}) \, \mathrm{d}\xi \\
& + \lambda_j^{(i)} \mathbf{\Psi}_0^{(i)} + \sum_{k=2}^{j-1} \lambda_k^{(i)} \boldsymbol{\phi}_{j-k}^{(1)} - \mathbf{g}_j^{(i)},
\end{aligned}$$

$$\mathbf{g}_j^{(i)} := \frac{1}{|\square|} \int\limits_{\square} \left( \mathcal{K}_{-1} \Phi_{j+1}^{(i)} + \mathcal{K}_0 \big( \Phi_{j+2}^{(i)} + (\Lambda_1 B(\partial_x) + \Lambda_0) \widetilde{\boldsymbol{\phi}}_{j-1}^{(i)} \big) + \sum_{k=1}^{N} S_{ik}^{(j-2)} \mathbf{\Upsilon}_2^{(k)} \right) \mathrm{d}\xi,$$

*being orthogonal to* $\mathbf{\Psi}_0^{(k)}$*,* $k = 1, \ldots, N$*, in* $L_2(\mathbb{R}^d; \mathbb{C}^n)$*. The numbers* $S_{ik}^{(j)}$ *and* $\lambda_j^{(i)}$ *are determined by*

$$S_{ik}^{(j)} = \frac{(\mathbf{g}_j^{(i)}, \mathbf{\Psi}_0^{(k)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}}{\tau_i - \tau_k}, \quad i \neq k, \qquad S_{ii}^{(j)} = 0, \qquad \lambda_j^{(i)} = (\mathbf{g}_j^{(i)}, \mathbf{\Psi}_0^{(i)})_{L_2(\mathbb{R}^d;\mathbb{C}^n)}.$$

Thus, the coefficients of the series (1.3), (1.5) are determined that completes the formal constructing of the asymptotics.

We denote

$$\lambda_{\varepsilon,k}^{(i)} := \lambda_0 + \sum_{j=1}^{k} \varepsilon^j \lambda_j^{(i)}, \quad \boldsymbol{\psi}_{\varepsilon,k}^{(i)}(x) := \mathbf{\Psi}_0^{(i)}(x) + \sum_{j=1}^{k} \varepsilon^j \mathbf{\Psi}_j^{(i)} \left( x, \frac{x}{\varepsilon} \right).$$

The next lemma follows directly from Lemma 3.2 and the equations (3.1).

**Lemma 3.3.** *The function* $\boldsymbol{\psi}_{\varepsilon,k}^{(i)} \in W_2^\infty(\mathbb{R}^d; \mathbb{C}^n)$ *solves the equation* $(\mathcal{H}_\varepsilon - \lambda_{\varepsilon,k}^{(i)}) \boldsymbol{\psi}_{\varepsilon,k}^{(i)} = \mathbf{f}_{\varepsilon,k}^{(i)}$*, where the function* $\mathbf{f}_{\varepsilon,k}^{(i)} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ *satisfies an uniform in* $\varepsilon$ *estimate*

$$\|\mathbf{f}_{\varepsilon,k}^{(i)}\|_{L_2(\mathbb{R}^d;\mathbb{C}^n)} \leqslant C \varepsilon^{k-1}.$$

Basing on Lemmas 2.3, 3.3 and proceeding completely in the similar way as in the proof of Lemma 4.3 and Theorem 1.1 in [13], one can show now that the eigenvalues $\lambda_\varepsilon^{(i)}$ and the associated eigenfunctions satisfy the asymptotic expansions (1.3), (1.5).

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